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N.A. Abiev[1] – *main author,*
A.K. Segizbaeva[2]

[1]Cand. Phys.-Math. Sci., Acc. Prof., [2]Master student

M.Kh.Dulaty Taraz Regional University,
Taraz, Republic of Kazakhstan
[1]abievn@mail.ru

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USING COMPUTER TECHNOLOGIES TO INVESTIGATION AND CONSTRUCTION PLANAR CURVES ASYMPTOTES

Abstract. Questions concerning the computer study of the existence of asymptotes of planar curves and their practical construction are considered. This need is due to routiness and laboriousness of manual computations and to automate the process. This problem especially concerns the case of an implicit equation of curves represented by high order polynomials. A modified version of the justification of the algorithm for constructing the asymptotes of a planar curves in the case of multiple roots of the leading term of such a polynomial is proposed. The algorithm is implemented as a computer program.

Keywords: planar curve, algebraic curve, polynomial of two variables, simple root, multiple root, asymptote.

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Introduction. Researchers often deal with high order algebraic curves defined by implicit functions. In particular, such problems are the subject of study not only in differential geometry, but also in algebraic geometry, dynamical systems and other branches of mathematics. In this paper we propose some simplified way to justify the algorithm for finding asymptotes of such curves and its computer implementation. Recall well known facts from [1,2], concerning asymptotes of planar curves.

Theorem 1. A straight line $Ax+By+C=0$ is an asymptote of a curve $x=\phi(t)$, $y=\psi(t)$ if and only if the following condition holds: $\lim_{t\to T}(A\phi(t)+B\psi(t)+C)=0$.

Theorem 1 provides a convenient tool for finding asymptotes. Without loss of generality, suppose that $Ax+By+C=0$ is not vertical, i.e. $B\neq 0$. Then

$$k=\lim_{t\to T}\frac{\psi(t)}{\phi(t)}=0, \qquad (1)$$

$$b=\lim_{t\to T}(\psi(t)-k\phi(t)). \qquad (2)$$

Vertical asymptotes can be found analogously.

The case of curves defined by implicit equation $F(x,y)=0$, presents certain difficulties. Consider such questions when $F(x,y)$ is a polynomial of degree n.

Theorem 2 [3]. Assume that a curve defined by $F(x,y)=0$. If the straigt line $y=kx+b$ is a non vertical asymptote of the curve $F(x,y)=0$, then the coefficient k satisfies the equation $F_n(1,k)=0$. The coefficient b which corresponds to such k, can be found

а) by the formula $b=\frac{F_{n-1}(1,k)}{F_n'(1,k)}$, if k is a simple root of the equation $F_n(1,k)=0$: $F_n(1,k)=0$ and $F_n'(1,k)\neq 0$;

б) from the equation $\frac{b^2}{2}F_n''(1,k)+bF_{n-1}'(1,k)+F_{n-2}(1,k)=0$, if k is a double root of the equation $F_n(1,k)=0$: $F_n(1,k)=F_n'(1,k)=0$, and $F_{n-1}(1,k)=0$;

в) there is no asymptote corresponding to such a multiple root k, if $F_{n-1}(1,k)\neq 0$.

Main results. We proposed a modified and effective version of proof theorem 2, different from the original one in [3]. Also we suggested a computer interpretation results of theorems 1 and 2 on concrete examples, confirming the advantage of computer tools over the manual labor.

We are going to exposition of our idea of proving Theorem 2. We will repeat reasoning of [3] until we start using the continuity property of polynomials of arbitrary degree and pass to limits in infinite sums. The equation of the curve $F(x,y)=0$ can be rewritten in the form

$$F(x,y)=F_n(x,y)+F_{n-1}(x,y)+...+F_0=0, \tag{3}$$

where $F_i(x,y)$ is a polynomial of degree i, homogeneous with respect to its arguments. Let us use the change of variables $\eta=\frac{y}{x},\ \xi=\frac{1}{x}$. Then we get an equation of our curve in new variables

$$F_n(1,\eta)+\xi F_{n-1}(1,\eta)+...+\xi^n F_0=0. \tag{4}$$

Assume that $F(x,y)=0$ has an asymptote defined by the equation $y=kx+b$. Then $k=\lim_{x\to\infty}\frac{y(x)}{x}=\lim_{\xi\to 0}\eta$.

As we noted above, further our idea is light different from ideas in [3], and strikingly different from ones in [2]. We propose offer a more concise way the equations we need, based on continuity of the function $F_n(x,y)$ as a polynomial. Namely, pass to limit in (4). Then

$$0=\lim_{\xi\to 0}(F_n(1,\eta)+\xi F_{n-1}(1,\eta)+...+\xi^n F_0)=\lim_{\xi\to 0}F_n(1,\eta)=F_n(1,\lim_{\xi\to 0}\eta)=F_n(1,k).$$

Thus we obtain the following equation, from which the coefficient k can be defined as its root:

$$F_n(1,k)=0. \tag{5}$$

For finding b consider a point $(x, y(x))$ on our curve, where $y(x)=kx+v$. In new coordinates: $\eta = k + v\xi$. Since $b=\lim_{x\to\infty}(y(x)-kx)=\lim_{x\to\infty} v$, this implies

$$b=\lim_{x\to\infty}\frac{y(x)/x-k}{1/x}=\lim_{\xi\to 0}\frac{\eta-k}{\xi}=\lim_{\xi\to 0} v. \tag{6}$$

Substitute the point $(x, kx+v)$, in other words, the point $(1, k+v\xi)$, into the equation of the curve $F_n(1,k+v\xi)+\xi F_{n-1}(1,k+v\xi)+...+\xi^n F_0=0$. After then each polynomial in this sum we expand to Taylor series. Clearly, such expansions contain finite number of summands. So

$$F_n(1,k)+v\xi F_n'(1,k)+\xi[F_{n-1}(1,k)+v\xi F_{n-1}'(1,k)]+O(\xi^2)=0. \tag{7}$$

Since $F_n(1,k)=0$, then $vF_n'(1,k)+F_{n-1}(1,k)+O(\xi)=0$.

The case of simple roots. Assume that the equation (5) admits only simple roots: $F_n'(1,k)\neq 0$. Passing to limit in the last equality as $\xi\to 0$ and taking into account (6), we obtain $bF_n'(1,k)+F_{n-1}(1,k)=0$. This yields

$$b=\frac{F_{n-1}(1,k)}{F_n'(1,k)}. \tag{8}$$

The case of multiple roots. For simplicity consider the case of double roots. Other cases are analogous. So let (5) to admit roots of multiplicity 2, i.e. $F_n'(1,k)=0$, but $F_n''(1,k)\neq 0$. Now the formula (8) is not valid. We have to consider high order terms in (7):

$$F_n(1,k)+v\xi F_n'(1,k)+$$
$$+\frac{(v\xi)^2}{2}F_n''(1,k)+\xi[F_{n-1}(1,k)+v\xi F_{n-1}'(1,k)]+\xi^2 F_{n-2}(1,k)+O(\xi^3)=0.$$

Using (5) and the condition $F_n'(1,k)=0$ we get

$$\frac{v^2\xi}{2}F_n''(1,k)+F_{n-1}(1,k)+v\xi F_{n-1}'(1,k)+\xi F_{n-2}(1,k)+O(\xi^2)=0.$$

It is necessary $F_{n-1}(1,k)=0$. Otherwise there is no asymptote. Indeed, $F_{n-1}(1,k)\neq 0$ would lead

$$0=\lim_{\xi\to 0}[\frac{v^2\xi}{2}F_n''(1,k)+F_{n-1}(1,k)+v\xi F_{n-1}'(1,k)+\xi F_{n-2}(1,k)+O(\xi^2)]=$$

$$=\lim_{\xi\to 0}F_{n-1}(1,k).$$

Hence, $\frac{v^2}{2}F_n''(1,k)+vF_{n-1}'(1,k)+F_{n-2}(1,k)+O(\xi)=0$.

Passing to limit as $\xi\to 0$ we obtain a quadratic equation to detect b:

$$\frac{b^2}{2}F_n''(1,k)+bF_{n-1}'(1,k)+F_{n-2}(1,k)=0. \qquad (9)$$

Computer interpretation theoretical results. All we need in Maple one can find in [4]. To demonstrate advantages of Maple consider the following curve from [1]:

$$\phi(t)=2t+3+\frac{1}{t-1},\quad \phi(t)=-t+2+\frac{4}{t-1} \qquad (10)$$

at $-\infty<t<\infty$. Obviously, at $t\to 1$, $t\to\infty$ the curve tends to infinite. Let us introduce (1) and (2). Then calculations on Maple give

$$A1 := 4x-19$$
$$A2 := -\frac{1}{2}x+\frac{7}{2}$$

Therefore, we have two asymptotes $y=4x-19$, $y=-0{,}5x+3{,}5$.
Graphs are:

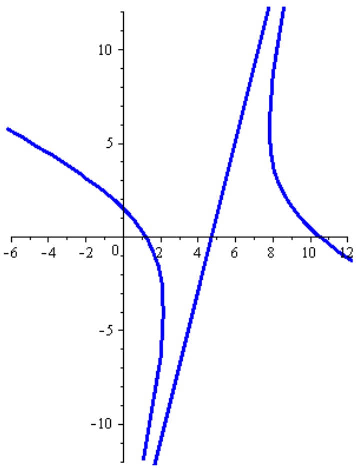

Fig. 1. The curve (10)

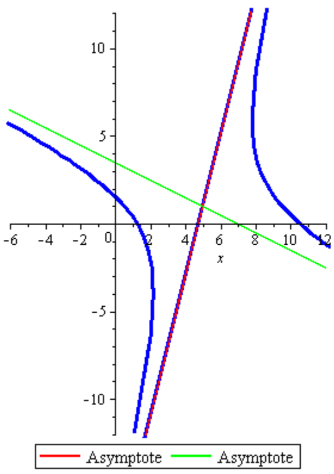

Fig. 2. The curve (10) and its asymptotes

It seems from Fig.2 that $y=4x-19$ intersects and even coincides with the middle part of the curve (10). But this is not so. From

$\psi(t)-(4x+19)=-9(t-1)$ we can see that the curve lies lower than the asymptote $y=4x-19$ at $t>1$ and the x-coordinate of a point on the curve tends to plus infinity at $t\to 1+0$. For $t<1$ the curve lies higher than the asymptote and $x\to-\infty$ as $t\to 1-0$.

Calculations on Maple also confirm, that (10) does not intersect the asymptote:

$> solve(\{\psi = k1\cdot\phi + b1\}, t);$

$\{t=1\}$

However $t=1$ does not belong to the domain of the equation

$\psi(t)-(4\phi(t)+19)=0$.

For interpretations in the implicit case consider the curve

$$F(x,y)=2ax^2y-xy^2+2a=0, \quad (11)$$

obtained in [5] as an isocline of a dynamical system related to the Ricci flow on generalizes Wallach spaces. Let $a=1/6$:

$> a := \frac{1}{6}:$

$> P := F(x,y);$

$P := \frac{1}{3}x^2y - xy^2 + \frac{1}{3}$

$> n := degree(P)$

$n := 3$

The combination of commands subs, expand and collect transforms the equation of the curve (11) into the form (4) and derives the equation (5):

$> \varphi := collect\left(subs\left(x=\frac{1}{\xi}, expand\left(\frac{subs(y=x\cdot\eta, P)}{x^n}\right)\right), \xi\right)$

$\varphi := \frac{1}{3}\eta - \eta^2 + \frac{1}{3}\xi^3$

$>$ **for** i **from** n **by** -1 **to** 0 **do** $F_i := coeff(\varphi, \xi, n-i)$ **end do**

$F_3 := \frac{1}{3}\eta - \eta^2$

$F_2 := 0$

$F_1 := 0$

$F_0 := \frac{1}{3}$

The following commands find roots of (5):

$> solo := solve(F_n, \eta)$

$solo := 0, \frac{1}{3}$

$> k1 := solo[1]; k2 := solo[2];$

$k1 := 0$

$k2 := \frac{1}{3}$

As you can see, the leading term of the polynomial (4) has two simple roots. Therefore, there exist two non vertical asymptotes.

The following commands find their equations:

$> subs\left(\eta = k1, b\cdot\frac{d}{d\eta}F_n + F_{n-1}\right)$

$\frac{1}{3}b$

$> b1 := solve(\%, b);$

$b1 := 0$

$> A1 := k1\cdot x + b1$

$A1 := 0$

$> subs\left(\eta = k2, b\cdot\frac{d}{d\eta}F_n + F_{n-1}\right)$

$-\frac{1}{3}b$

$> b2 := solve(\%, b);$

$b2 := 0$

$> A2 := k2\cdot x + b2$

$A2 := \frac{1}{3}x$

So computer found two asymptotes $y = 0$, $y = \frac{1}{3}x$.

There exist a vertical asymptote $x = 0$ as well, because of

$> \psi := subs\left(y = \frac{1}{\omega}, expand\left(\frac{subs(x = y\cdot\theta, P)}{y^d}\right)\right)$

$\psi := \frac{1}{3}\theta^2 - \theta + \frac{1}{3}\omega^3$

$> solo := solve(coeff(\psi, \omega, 0), \theta)$

$solo := 0, 3$

The graphs are:

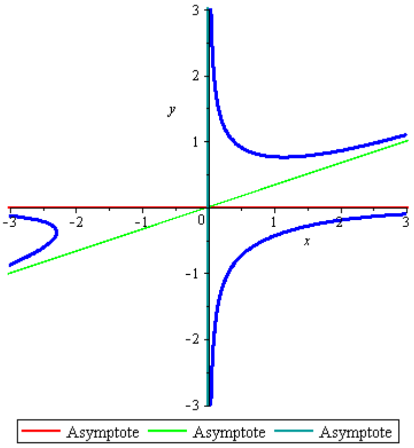

Fig. 3. The curve (11) and its asymptotes. The case when (5) has simple roots

Consider now a curve from [2]:

$$y^3 - 4x^2 y + 2x^2 + y^2 - 5x + y + 4 = 0, \qquad (12)$$

Introducing initial data

$P := -4x^2 y + y^3 + 2x^2 + y^2 - 5x + y + 4$
$n := 3$
Since commands are repeating, we show only results:
$\varphi := 4\xi^3 + (\eta - 5)\xi^2 + (\eta^2 + 2)\xi + \eta^3 - 4\eta$
$F_3 := \eta^3 - 4\eta$
$k1 := 0 \;\; k2 := 2 \;\; k3 := -2$

The roots are simple in this case too:

$A1 := \frac{1}{2}$

$A2 := 2x - \frac{3}{4}$

$A3 := -2x - \frac{3}{4}$

So Maple found asymptotes $y = \frac{1}{2}$, $y = \pm 2x - \frac{3}{4}$. There is no vertical asymptote**:**

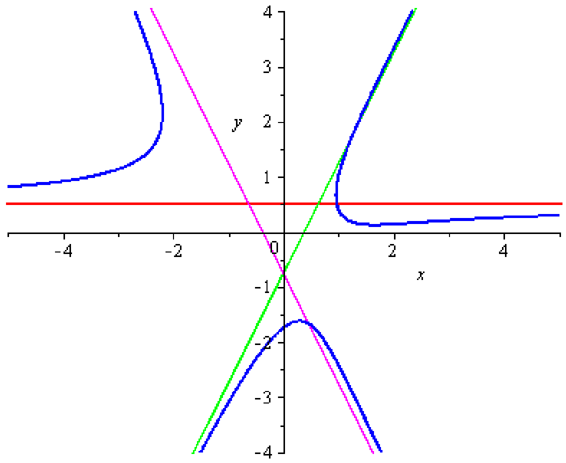

Fig. 4. The curve (12) and its asymptotes. The case when (5) has simple roots

Asymptotes do not meet the curve only at infinity. But on a bounded segment of variation of the argument they can intersect. Below we show such points depicted on graphs by circles.

$solve\left(\left\{y = -2x - \frac{3}{4}, P\right\}, \{x, y\}\right); x0 := \frac{217}{472}; evalf(\%); y0 := -\frac{197}{118}; evalf(\%);$

$x0 := \frac{217}{472} \quad y0 := -\frac{197}{118}$

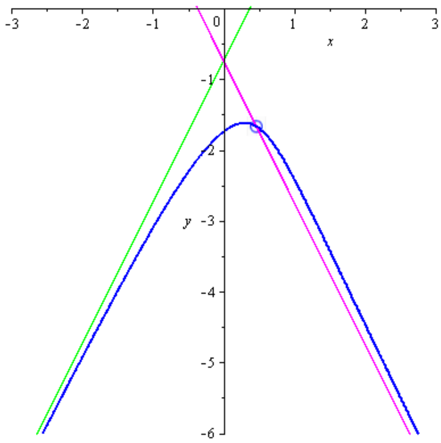

Fig. 5. Intersection (12) with the asymptote $y=-2x-\frac{3}{4}$.

\>

$solve\left(\left\{y=2x-\frac{3}{4}, P\right\}, \{x, y\}\right); x0 := \frac{31}{24}; evalf(\%); y0 := \frac{11}{6}; evalf(\%);$

$x0 := \frac{31}{24} \quad y0 := \frac{11}{6}$

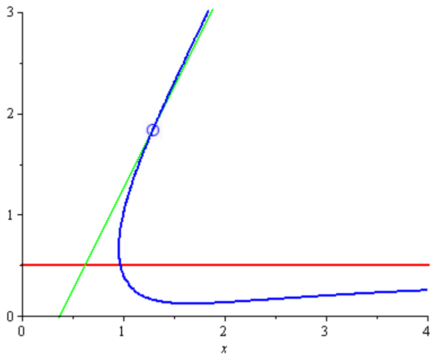

Fig. 6. Intersection (12) with the asymptote $y=2x-\frac{3}{4}$

Consider now a curve

$$x^3+3x^2y-4y^3-x+y+3=0. \quad (13)$$

Calculations give:

$k1 := 1 \quad k2 := -\frac{1}{2} \quad k3 := -\frac{1}{2}$

We are in the case of multiple roots.
Using (8) for the simple root $\eta=1$ we get

$A1 := x$

Hence, one of the asymptotes is: $y=x$.

For the multiple root $\eta=-\frac{1}{2}$ the formula (8) is not valid, so we have to find b from the equation (9).

Checking the condition $F_{n-1}(1,k)=0$ shows existence of asymptotes

\>
if $subs\left(\eta = k2, F_{n-1}\right) = 0$ **then** *print*("There exists an asymptote") **end if**

"There exists an asymptote"

Therefore they can be found from (9). The corresponding Maple commands and graphs:

\> $subs\left(\eta = k2, \frac{b^2}{2}\cdot\frac{d^2}{d\eta^2}F_n + b\cdot\frac{d}{d\eta}F_{n-1} + F_{n-2}\right)$

$6b^2 - \frac{3}{2}$

\> $b2 := solve(\%, b);$

$b2 := \frac{1}{2}, -\frac{1}{2}$

\> $A2 := k2\cdot x + b2[1]$

$A2 := -\frac{1}{2}x + \frac{1}{2}$

\> $A3 := k2\cdot x + b2[2]$

$A3 := -\frac{1}{2}x - \frac{1}{2}$

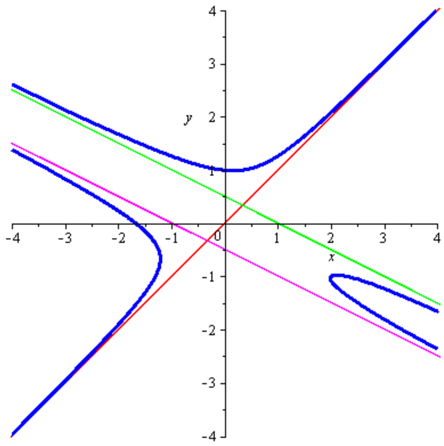

Fig. 7. The curve (13) and its asymptotes. The case when (5) has multiple roots

Conclusion. We have proposed a modified and more efficient version of the proof of Theorem 2 concerning asymptotes of high-order curves defined by implicit functions. We also showed the possibility of using computer to perform routine computations. As noted the case of parametric curves does not present any difficulty having a simple mathematical apparatus (1) and (2), which is very easy to program.

Much more complicated is the case of curves defined implicitly by high-order polynomials. To demonstrate our main idea we restricted ourselves considering the case of double roots, where the computer performs expansions to Taylor series and derivations of corresponding equations. The idea can be extended to the general case. The proposed program also does not claim to be complete. For example, it does not provide for a procedure of determining the multiplicity of polynomial's roots and the consequent choice between formulas (8) and (9). While this choice is made by the user himself after the computer program makes up a complicated polynomial equation $F_n(1,\eta)=0$ and finds its roots. So the program can be further improved.

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Н.А. Абиев, А.Қ. Сегізбаева

М.Х. Дулати атындағы Тараз өңірлік университеті, Тараз, Қазақстан

ЖАЗЫҚ ҚИСЫҚТАРДЫҢ АСИМПТОТАЛАРЫН ЗЕРТТЕУ МЕН ҚҰРУҒА КОМПЬЮТЕРЛІК ТЕХНОЛОГИЯЛАРДЫ ПАЙДАЛАНУ

Аңдатпа. Жұмыста жазық қисықтардың асимптоталарының болуын компьютерде зерттеу және оларды практикада құру сұрақтары қарастырылады. Жұмыстың маңыздылығы қолмен есептеулердің күрделілігі мен құру үрдісін автоматтандыру қажеттілігінен туады. Әсіресе мұндай проблема қисықтардың жоғары дәрежелі көпмүшелермен айқын емес түрде берілетін жағдайына қатысты зерттеледі. Жұмыста осындай көпмүшенің бас мүшесінің еселі түбірлері болған жағдайындағы асимптота құру алгоритмін негіздеудің модификацияланған нұсқасы ұсынылады. Алгоритм компьютерлік бағдарлама түрінде жүзеге асырылған.

Тірек сөздер: жазық қисық, алгебралық қисық, екі айнымалылы көпмүше, қарапайым түбір, еселі түбір, асимптота.

Н.А. Абиев, А.К. Сегизбаева

Таразский региональный университет им. М.Х. Дулати, Тараз, Казахстан

ИСПОЛЬЗОВАНИЕ КОМПЬЮТЕРНЫХ ТЕХНОЛОГИЙ К ИССЛЕДОВАНИЮ И ПОСТРОЕНИЮ АСИМПТОТ ПЛОСКИХ КРИВЫХ

Аннотация. В работе рассматриваются вопросы компьютерного исследования существования асимптот плоских кривых и их практического построения. Важность работы вызвана рутинностью и трудоемкостью ручных вычислений и необходимостью автоматизации процесса построения. Особенно эта проблема касается случая неявного уравнения кривых, представленного многочленами высокого порядка. В работе предложена модифицированная версия обоснования алгоритма построения асимптот плоской кривой в случае кратных корней ведущего члена такого многочлена. Алгоритм реализован в виде компьютерной программы.

Ключевые слова: плоская кривая, алгебраическая кривая, многочлен от двух переменных, простой корень, кратный корень, асимптота.